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OCCUPATION TIME LAWS FOR BIRTH AND DEATH PROCESSES

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1. Introduction

Let $X(t)$ with $t \geqq 0$ be a (Borel) measurable stationary Markov process whose state space is a metric space $\mathcal{E}$ and whose transition probability function is

$$P(t; x, E) = P\{X(t+s) \in E | X(s) = x\}. \tag{1}$$

Darling and Kac [3] studied the limiting distribution of the random variables

$$Z(t) = \int_0^t V[X(\tau)]\, d\tau \tag{2}$$

as $t \to \infty$, where V is a nonnegative measurable function. If V is the characteristic function of a set E then $Z(t)$ is the occupation time of E.

Darling and Kac assume that

$$\int_0^\infty e^{-st} P(t; x, E)\, dt = \pi(E) h(s) + h_1(s; x, E), \tag{3}$$

where $h(s) \to \infty$ as $s \to 0+$ and

$$\lim_{s \to 0+} \frac{1}{h(s)} \int h_1(s; x, dy) V(y) = 0, \tag{4}$$

the convergence in (4) being uniform on the set $\{x; V(x) > 0\}$. They then show that the kth moment $\mu_k(t)$ of $Z(t)$ satisfies

$$s \int_0^\infty e^{-st} \mu_k(t)\, dt \sim k![\pi(E) h(s)]^k, \qquad s \to 0+. \tag{5}$$

Under the additional hypothesis

$$h(s) = s^{-\alpha} L\left(\frac{1}{s}\right), \tag{6}$$

where $0 \leqq \alpha < 1$ and $L(1/s)$ is slowly varying (see below) as $s \to 0+$, they deduce from Karamata's Tauberian theorem that

$$\lim_{t \to \infty} P\left\{\frac{Z(t)}{\pi(E) h(1/t)} \leqq u\right\} = G_\alpha(u), \tag{7}$$

where G_α denotes the Mittag-Leffler distribution,

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$$G_\alpha(u) = \frac{1}{\pi\alpha}\int_0^u \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}\Gamma(\alpha n + 1)y^{n-1}\sin n\pi\alpha\, dy \tag{8}$$

whose kth moment is $k!/\Gamma(\alpha k + 1)$. For $\alpha = 0$ this becomes $G_0(u) = 1 - \exp(-u)$ while for $\alpha = 1/2$ we have $G_{1/2}(u) = \pi^{-1/2}\int_0^u \exp(-y^2/4)\, dy$.

Darling and Kac also establish a remarkable converse theorem. They show that if (3) and (4) are satisfied and if for some normalizing function $v(t) > 0$

$$\lim_{t\to\infty} P\left\{\frac{Z(t)}{v(t)} \leqq u\right\} = G(u), \tag{9}$$

where G is a nondegenerate distribution, then $h(s)$ satisfies (6) and hence $G(x) = G_\alpha(cx)$ for some constant c.

Only in a few exceptional situations are sufficient conditions on the process known which insure the hypotheses (3), (4), and (6). Darling and Kac discuss as examples the Wiener process and a class of discrete time processes which are sums of identically distributed independent random variables.

Our primary purpose here is to discuss the validity of (3), (4), and (6) for birth and death processes (described below). Of course these hypotheses do not hold for arbitrary birth and death processes, and what is required is to find natural assumptions concerning the given data of a birth and death process from which one can deduce the hypotheses. Now in problems involving birth and death processes the data one is ordinarily given consists of the birth and death rates λ_n, μ_n. We present a theory in which the hypotheses (3), (4), and (6) are derived from knowledge of the asymptotic behavior of the birth and death rates λ_n, μ_n as $n \to \infty$. Both the methods and the results can be extended to general diffusion processes and random walks.

In section 3 this result is extended to more complicated cases involving slowly varying functions.

Since the appearance of the work of Darling and Kac, it has been found that once the hypotheses (3), (4), and (6) are known it is possible to deduce limit laws not only for the occupation times $Z(t)$ but also for numerous other random variables associated with the process. Notable contributions along these lines have been made by Dynkin [4], Lamperti [11], [12], and Takács [14] (see the bibliography in [14]). Some of the results of these authors are discussed below and applications are made to birth and death processes.

We also show that for birth and death processes related assumptions lead to limit laws for the important maximum random variables

$$M(t) = \max_{0\leqq\tau\leqq t} X(\tau). \tag{10}$$

This result is both interesting in connection with various applications and also in that it serves to indicate the broader significance of our main problem.

We now describe the main problem in greater detail. A birth and death process is a stationary Markov process whose state space $\mathcal{E}$ is the nonnegative integers and whose transition probability matrix

$$P_{ij}(t) = P\{X(t+s) = j \mid X(s) = i\} \tag{11}$$

satisfies the conditions, as $t \to 0$,

$$P_{ij}(t) = \begin{cases} \lambda_i t + o(t) & \text{if } j = i+1, \\ \mu_i t + o(t) & \text{if } j = i-1, \\ 1 - (\lambda_i + \mu_i)t + o(t) & \text{if } j = i, \end{cases} \tag{12}$$

where $\lambda_i > 0$ for $i \geqq 0$, $\mu_i > 0$ for $i \geqq 1$, and $\mu_0 \geqq 0$. In all the cases treated below we assume $\mu_0 = 0$.

Associated with the process is a system $\{Q_n(x)\}$ of polynomials defined by the recurrence formulas

$$\begin{gathered} Q_{-1}(x) \equiv 0, \qquad Q_0(x) = 1, \\ -xQ_n(x) = -(\lambda_n + \mu_n)Q_n(x) + \lambda_n Q_{n+1}(x) + \mu_n Q_{n-1}(x), \qquad n \geqq 0. \end{gathered} \tag{13}$$

Clearly $Q_n(0) = 1$ for every n. It is shown in [7] that there is at least one positive regular measure ψ on $0 \leqq x < \infty$ such that

$$\int_0^\infty Q_i(x)Q_j(x)\, d\psi(x) = \frac{\delta_{ij}}{\pi_j} \qquad i, j = 0, 1, 2, \cdots, \tag{14}$$

where

$$\pi_0 = 1, \qquad \pi_j = \frac{\lambda_0\lambda_1 \cdots \lambda_{j-1}}{\mu_1\mu_2 \cdots \mu_j} \qquad \text{for } j \geqq 1. \tag{15}$$

The measure ψ is called the spectral measure of the process. In our present cases the ψ measure will be uniquely determined by the parameters $\{\lambda_n\}$ and $\{\mu_n\}$. This applies to the bulk of birth and death processes of physical and statistical significance. The transition matrix $P(t)$ is represented by the formula

$$P_{ij}(t) = \pi_j \int_0^\infty e^{-xt} Q_i(x)Q_j(x)\, d\psi(x). \tag{16}$$

For a full discussion of the properties of birth and death processes we refer the reader to [7], [8].

In section 1 we establish the identifications

$$h(s) = \int \frac{d\psi(x)}{x+s}, \qquad \pi(E) = \sum_{j \in E} \pi_j. \tag{17}$$

The condition $h(s) \to \infty$ as $s \to 0+$ is satisfied if and only if the process is recurrent or equivalently the series $\sum 1/\lambda_n\pi_n$ diverges. When this is the case condition (4) is easily shown to be satisfied if V is the characteristic function of a finite set, and other cases can be analyzed by routine methods. The main part of the work lies therefore in showing that $h(s)$ is of the form (6).

The function $h(s)$ is expressed, rather indirectly, in terms of the parameters $\{\lambda_n\}$, $\{\mu_n\}$ by the important formula ([7], p. 529)

$$\int_0^\infty \frac{d\psi(x)}{x+s} = \sum_{n=0}^\infty \frac{1}{\lambda_n\pi_n Q_n(-s)Q_{n+1}(-s)}. \tag{18}$$

All quantities in the right member of this expression are determined directly from the parameters by the definition of π_n and by the recurrence formula (13). In section 3 we make a detailed analysis of this formula and show that the asymptotic behavior of $h(s)$ as $s \to 0+$ can be deduced, under suitable conditions, from the asymptotic behavior of the parameters for large n. In this way we find quite general sufficient conditions for $h(s)$ to be of the form (6), as follows.

THEOREM 1. *If as* $n \to \infty$

$$\frac{1}{\lambda_n \pi_n} \sim C n^{\beta-1}, \qquad \pi_n \sim D n^{\gamma-1}, \tag{19}$$

where C, D, β, γ *are positive constants, then as* $s \to 0+$

$$h(s) \sim H s^{-\alpha}, \tag{20}$$

where

$$\alpha = \frac{\beta}{\beta+\gamma}, \qquad H = \frac{(\beta+\gamma)^{2\alpha-1}}{C^{\alpha-1}D^{\alpha}} \frac{\Gamma(\alpha)}{\Gamma(1-\alpha)}. \tag{21}$$

In section 2 we also prove the following. Note that the assumptions are weaker than those required by theorem 1.

THEOREM 2. *If as* $n \to \infty$

$$\frac{1}{\lambda_n \pi_n} \sim C n^{\beta-1}, \qquad \pi_n \sim D n^{\gamma-1}, \tag{22}$$

where C, D, γ, $\beta + \gamma$ *are positive constants, then*

$$\lim_{n \to \infty} Q_n\left(\frac{-s}{n^{\beta+\gamma}}\right) = I(s), \tag{23}$$

where

$$I(s) = \Gamma(1-\alpha) \sum_{r=0}^{\infty} \frac{1}{r!\Gamma(r-\alpha+1)} \left[\frac{CDs}{(\beta+\gamma)^2}\right]^r \tag{24}$$

and $\alpha = \beta/(\beta+\gamma)$. *The convergence is uniform in every bounded region of the complex variable* s.

Theorem 2 is exploited in section 4 to obtain limit laws for the maximum random variables.

We conclude the general discussion by indicating the connection of the asymptotic relationship (6) and the rate of decay of the tails of the first passage time distributions of the process.

The condition (6) can be translated into an equivalent condition on the tails of the first return time distribution $F_{00}(t)$ of the zero state. The formula of equation (1.3) in [8] in conjunction with (16) yields

$$s \int e^{-st}[1 - F_{00}(t)]\, dt = \frac{1}{(\lambda_0 + s)h(s)}. \tag{25}$$

Since $1 - F_{00}(t)$ is decreasing, it follows from Karamata's theorem that

$$1 - F_{00}(t) \sim \frac{1}{\lambda_0 L(t) t^\alpha \Gamma(1-\alpha)} \qquad \text{as} \quad t \to \infty \tag{26}$$

if and only if, similar to (6),

$$h(s) \sim s^{-\alpha} L\left(\frac{1}{s}\right), \qquad 0 < \alpha < 1, \quad \text{as} \quad s \to 0+, \tag{27}$$

where $L(u)$ is a slowly varying function, $u \to \infty$.

In a similar manner we obtain

$$1 - F_{ii}(t) \sim \frac{1}{\lambda_i \pi_i \Gamma(1-\alpha) t^\alpha L(t)} \qquad \text{as} \quad t \to \infty, \tag{28}$$

where $F_{ii}(t)$ denotes the distribution of the first return time to the state i.

We note for later reference that the asymptotic relation (26) is the necessary and sufficient condition that a distribution function of a positive random variable belongs to the domain of attraction of a stable law of index α, $0 < \alpha < 1$ (see [5]). In this connection Dynkin [4] and independently Lamperti [11] developed the following class of limit theorems. Let

$$S_0 = 0, \qquad S_n = \xi_1 + \xi_2 + \cdots + \xi_n, \qquad n \geqq 1, \tag{29}$$

where ξ_i are independent identically distributed positive random variables whose distribution function is $F(u)$. Define the integer-valued random variables $\nu(t)$ by

$$\nu(t) = n \quad \text{if} \quad S_n \leqq t < S_{n+1}, \qquad t > 0, \tag{30}$$

and set $\gamma(t) = t - S_{\nu(t)}$ and $\delta(t) = S_{\nu(t)+1} - t$. They prove that the limit laws

$$\begin{aligned} \lim_{t\to\infty} P\left\{\frac{\gamma(t)}{t} \leqq x\right\} &= \frac{\sin \alpha\pi}{\pi} \int_0^x u^{\alpha-1}(1-u)^{-\alpha}\,du, \qquad 0 < x < 1, \\ \lim_{t\to\infty} P\left\{\frac{\delta(t)}{t} \leqq x\right\} &= \frac{\sin \pi\alpha}{\pi} \int_0^x u^{-\alpha}(1+u)^{-1}\,du, \qquad 0 < x < \infty, \end{aligned} \tag{31}$$

hold if and only if $F(u)$ satisfies (26). They also show that when $E(\xi) = \infty$ the random variables $\gamma(t)$ and $\delta(t)$ cannot be normalized in any fashion to produce a nondegenerate limit law other than that of (31). (Trivial modifications of the normalizing constants like any function asymptotic to ct are considered not different.) When $E(\xi) < \infty$, the distributions of $\gamma(t)$ and $\delta(t)$ converge without renormalization. This is a familiar fact of renewal theory [13].

We shall apply the conclusions (31) to the random variable $Y(t)$ which is the time since the last visit to a fixed state (say 0) of the birth and death process. In particular, under the hypothesis (63) given below we obtain (6) or equivalently (26). It is clear that we are in the situation of (29) where ξ_i are distributed according to $F_{00}(t)$ and the relevant identification is $Y(t) = \gamma(t)$.

Similar results can easily be obtained for random walks by using the substitutions explained in [9], section 2.

2. Preliminary formulas

The existence of a representation

$$\int_0^\infty e^{-st}P_{ij}(t)\,dt = \pi(j)h(s) + h_1(s;i,j) \tag{32}$$

for the Laplace transform of the transition matrix of a recurrent birth and death process, where $h(s) \to +\infty$ and $h_1(s;i,j)/h(s) \to 0$ as $s \to 0+$, may be deduced in two ways, which we now sketch. Observe that if the process is not recurrent then the integrals $\int_0^\infty P_{ii}(t)\,dt$ are finite and there can be no such representation.

Our first method appeals to the results of section 7 in [7] where it is shown that

$$\int_0^\infty e^{-st}P_{ij}(t)\,dt = Q_i^{(j)}(-s) + Q_i(-s)\left[Q_j^{(0)}(-s) + Q_j(-s)\int_0^\infty \frac{d\psi(x)}{x+s}\right]\pi_j, \tag{33}$$

where π_j, $Q_j(\cdot)$ are the constants and polynomials defined above and $Q_i^{(j)}(\cdot)$ are the polynomials

$$Q_i^{(j)}(x) = \pi_j \int_0^\infty Q_j(y)\,\frac{Q_i(x) - Q_i(y)}{x - y}\,d\psi(y). \tag{34}$$

Since the process is recurrent the integral $\int_0^\infty d\psi/x$ diverges [8] while on the other hand $Q_j(0) = 1$ so $\lim_{s\to 0+} [1 - Q_j(-s)] \int [1/(x+s)]\,d\psi$ is finite. Hence, rearranging (33) we have the desired representation with

$$\pi(j) = \pi_j, \qquad h(s) = \int_0^\infty \frac{d\psi(x)}{x+s}. \tag{35}$$

Our second method gives the result in greater generality. We consider a recurrent irreducible Markov chain with stable states and transition matrix $P_{ij}(t)$. Let $q_0 = -P'_{00}(0)$ and ${}_iP_{ij}(t)$ be the probability the particle is at j at time t having started at i and without ever returning to i after leaving i. If

$$h(s) = \int_0^\infty e^{-st}P_{00}(t)\,dt, \tag{36}$$

then $h(s) \to +\infty$ as $s \to 0+$ if and only if the process is recurrent. Using results of Chung [1], [2] and Abelian arguments we find a representation of the desired form with the above $h(s)$ and $\pi(j) = {}_0P^*_{0j}q_0$, where ${}_iP^*_{ij} = \int_0^\infty {}_iP_{ij}(t)\,dt$. Specializing this to a recurrent birth and death process gives the previous result again. The details are as follows,

It is proved in [1] that if the process is recurrent, then ${}_0P^*_{0j} < \infty$. Observe that

$$(37)\qquad \int_0^\infty e^{-st}P_{ij}(t)\,dt = s\int_0^\infty e^{-su}\left(\int_0^u P_{ij}(t)\,dt\right)du$$

$$= s\int_0^\infty e^{-su}\left(\int_0^u P_{jj}(t)\,dt\right)du + s\int_0^\infty e^{-su}\left[\frac{\int_0^u P_{ij}(\xi)\,d\xi}{\int_0^u P_{jj}(\xi)\,d\xi} - 1\right]\left(\int_0^u P_{jj}(\xi)\,d\xi\right)du$$

$$= A_j(s) + \bar{h}(s, i, j).$$

Recurrence of the process implies

$$(38)\qquad \begin{aligned} &\lim_{u\to\infty}\int_0^u P_{jj}(t)\,dt = \infty, \qquad \text{for any } j,\\ &\lim_{u\to\infty}\frac{\int_0^u P_{ij}(t)\,dt}{\int_0^u P_{jj}(t)\,dt} = F_{ij}^* = 1\end{aligned}$$

(see Chung [1] and [2]). Here F_{ij}^* is the probability of reaching j from i in finite time.

A standard Abelian argument leads to the conclusion $\lim_{s\to 0} \bar{h}(s, i, j)/A_j(s) = 0$. In fact, choose M so large that

$$(39)\qquad \left|\frac{\int_0^u P_{ij}(\xi)\,d\xi}{\int_0^u P_{jj}(\xi)\,d\xi} - 1\right| < \epsilon \qquad \text{for} \quad u > M.$$

Then in view of (38), we obtain

$$(40)\qquad \overline{\lim_{s\to 0}}\frac{\bar{h}(s, i, j)}{A_j(s)} \leqq \epsilon$$

and the assertion is established. But

$$(41)\qquad A_j(s) = \pi_j h(s) + h^*(s, j),$$

where π_j and $h(s)$ are defined in (35) and $h^*(s, j)$ is determined in the obvious manner, namely

$$(42)\qquad h^*(s, j) = s\int_0^\infty e^{-su}\left(\int_0^u [P_{jj}(t) - P_{00}(t)]\,dt\right)du.$$

It is known that if the process is recurrent, then ${}_0P_{oj}^* < \infty$ and

$$(43)\qquad \lim_{u\to\infty}\frac{\int_0^u P_{jj}(\xi)\,d\xi}{\int_0^u P_{00}(\xi)\,d\xi} = {}_0P_{oj}q_0$$

(see Chung [1]). As previously it follows that $h^*(s, j)/h(s) \to 0$ when $s \to 0+$. Letting $h_1(s, i, j) = h^*(s, j) + \bar{h}(s, i, j)$, the decomposition is exhibited.

A recurrent birth and death process is either *ergodic*, in which case ψ has positive mass ρ at the origin [8] and $h(s) \sim \rho/s$ or else the process is *recurrent null*, in which case ψ has no mass at the origin and $h(s) \to \infty$, $sh(s) \to 0$ as $s \to 0+$.

In order for $h(s)$ to be of the form (6) with $0 \leqq \alpha < 1$ the process must be recurrent null, or equivalently [8] $\sum \pi_n = +\infty$, $\sum 1/\lambda_n\pi_n = +\infty$.

The plan of section 2 is to assume $1/\lambda_n\pi_n \sim Cn^{\beta-1}$, $\pi_n \sim Dn^{\gamma-1}$, where $0 < \beta, \gamma$, and find the asymptotic behavior of $h(s)$ from formula (18).

In preparation for this we express the coefficients of the polynomials $Q_n(x)$ in terms of the basic quantities π_n, $1/\lambda_n\pi_n$. First the recurrence relation is written in the form

$$\begin{aligned} -x\pi_nQ_n(x) &= \lambda_n\pi_n[Q_{n+1}(x) - Q_n(x)] - \lambda_{n-1}\pi_{n-1}[Q_n(x) - Q_{n-1}(x)], \qquad n \geqq 1, \\ -x\pi_0Q_0(x) &= \lambda_0\pi_0[Q_1(x) - Q_0(x)]. \end{aligned} \tag{44}$$

A first summation, and then another summation, produce

$$\begin{aligned} -x\sum_{k=0}^{n} \pi_kQ_k(x) &= \lambda_n\pi_n[Q_{n+1}(x) - Q_n(x)], \qquad n \geqq 0, \\ Q_n(x) &= 1 - x\sum_{k=0}^{n-1}\frac{1}{\lambda_k\pi_k}\sum_{l=0}^{k}\pi_lQ_l(x), \qquad n \geqq 1. \end{aligned} \tag{45}$$

Repeated differentiation of this identity using the polynomial character of Q_k leads to

$$\frac{Q_n^{(r)}(0)}{r!} = -\sum_{k=r-1}^{n-1}\frac{1}{\lambda_k\pi_k}\sum_{l=r-1}^{k}\frac{\pi_lQ_l^{(r-1)}(0)}{(r-1)!}, \qquad n \geqq r \geqq 1. \tag{46}$$

We may iterate this relationship to get the explicit expression

$$\frac{Q_n^{(r)}(0)}{r!} = (-1)^r\sum_{k=r-1}^{n-1}\frac{1}{\lambda_k\pi_k}\sum_{l=r-1}^{k}\pi_l\sum_{j=r-2}^{l-1}\frac{1}{\lambda_j\pi_j}\sum_{i=r-2}^{j}\pi_i\cdots\sum_{\nu=0}^{s-1}\frac{1}{\lambda_\nu\pi_\nu}\sum_{\mu=0}^{\nu}\pi_\mu. \tag{47}$$

We are now ready to determine the asymptotic behavior of (35) as $s \to 0+$. It is instructive to begin with the analysis of a special case. We will use first the crude inequalities

$$1 + \omega_ns \leqq Q_n(-s) \leqq e^{\omega_ns}, \qquad s > 0, \tag{48}$$

where

$$\omega_n = \sum_{k=0}^{n-1}\frac{1}{\lambda_k\pi_k}\sum_{l=0}^{k}\pi_l = -Q_n'(0). \tag{49}$$

The left-hand inequality is immediate since all the coefficients of $Q_n(-s)$ are strictly positive. The right-hand inequality is a simple consequence of the fact that $Q_n(x)$ has only positive roots and $Q_n(0) = 1$. Indeed,

$$Q_n(-s) = \prod_{i=1}^{n}\left(1 + \frac{s}{a_{n,i}}\right) < \exp\left(s\sum_{i=1}^{n}\frac{1}{a_{n,i}}\right) = e^{s\omega_n}. \tag{50}$$

It follows from (48) and (18) that

$$\sum_{n=0}^{\infty}\frac{1}{\lambda_n\pi_n}\exp[-s(\omega_n + \omega_{n+1})] \leqq h(s) \leqq \sum_{n=0}^{\infty}\frac{1}{\lambda_n\pi_n[1 + (\omega_n + \omega_{n+1})s]}. \tag{51}$$

In some special cases these crude bounds can be used to secure the precise asymptotic growth of $h(s)$. An example is hypothesis A below.

HYPOTHESIS A. $\pi_n \sim D$, $1/\lambda_n\pi_n \sim C/n$.

Then trivially, $\omega_n \sim CDn$ and $\omega_n + \omega_{n+1} \sim 2CDn$. We define

$$\theta(x) = \sum_{k \leqq x} \frac{1}{\lambda_k \pi_k}. \tag{52}$$

It follows easily that the asymptotic growth of

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n\pi_n[1 + (\omega_n + \omega_{n+1})s]}, \qquad s \to 0+, \tag{53}$$

coincides with that of

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n\pi_n[1 + Kns]} = \int_0^{\infty} \frac{d\theta(x)}{[1 + Kxs]}, \qquad K = 2CD. \tag{54}$$

Let $Ks = 1/\lambda$, then we seek to determine the behavior of

$$\lambda \int_0^{\infty} \frac{d\theta(x)}{\lambda + x}, \qquad \lambda \to \infty, \tag{55}$$

where $\theta(x) \sim C \log x$ as $x \to \infty$. A standard Abelian argument applied to the Stieltjes transform shows that

$$\lambda \int \frac{d\theta(x)}{\lambda + x} \sim C \log \lambda \qquad \text{as} \quad \lambda \to \infty. \tag{56}$$

Hence

$$\int \frac{d\theta(x)}{(1 + Kxs)} \sim C \log \frac{1}{s} \qquad \text{as} \quad s \to 0+ \tag{57}$$

and the constant C is that appearing in the statement of hypothesis A. The details providing the verification of (57) constitute a simpler version of the type of reasoning embodied in the proof of lemma 3 below and will therefore be omitted.

We find the asymptotic growth of $\sum_{n=0}^{\infty} (1/\lambda_n\pi_n) \exp[-s(\omega_n + \omega_{n+1})]$ agrees with that of the Laplace transform $\int_0^{\infty} e^{-Ksx} d\theta(x)$, using a similar method. An Abelian argument for the Laplace transform shows that

$$\int_0^{\infty} e^{-Ksx}\, d\theta(x) \sim C \log \frac{1}{s} \qquad \text{as} \quad s \to 0+. \tag{58}$$

It is important to notice that the same constant C appears in both (57) and (58). The inequalities (51) compared with (57) and (58) compel the limit relation

$$\lim_{s \to 0+} \frac{1}{\log (1/s)} h(s) = C. \tag{59}$$

In summary, *if*

$$\frac{1}{\lambda_n\pi_n} \sim \frac{C}{n} \quad \textit{and} \quad \pi_n \sim D, \qquad \textit{then} \quad h(s) \sim C \log \frac{1}{s} \quad \textit{as} \quad s \to 0+. \tag{60}$$

We consider next the asymptotic properties of $h(s)$ in the case where

$$\frac{1}{\lambda_n \pi_n} \sim C n^{\beta-1} \quad \text{and} \quad \pi_n \sim D n^{\gamma-1}, \qquad 0 < \beta, \gamma, \tag{61}$$

and try to imitate the above technique. (Note that hypothesis A is not covered by the present assumptions.) An analysis of the corresponding Laplace and Stieltjes transforms leads to the limit relations

$$\begin{aligned} \lim_{s \to 0+} s^{\beta/(\beta+\gamma)} \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n} \exp\left[-(\omega_n + \omega_{n+1})s\right] &= C_1, \\ \lim_{s \to 0+} s^{\beta/(\beta+\gamma)} \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n [1 + (\omega_n + \omega_{n+1})s]} &= C_2. \end{aligned} \tag{62}$$

Unfortunately, we now have $0 < C_1 < C_2$. This means that the bounds furnished indicate only the order of magnitude of $h(s)$ for $s \to 0+$, but the asymptotic limit is not established. One should mention at this point that the Darling-Kac limit theorems require the existence of an actual asymptotic limit To establish the existence of the limit of $s^{\beta/(\beta+\gamma)} h(s)$ as $s \to 0+$ it is necessary to use considerable care and ingenuity in approximating the polynomials $Q_n(-s)$ for s near zero. This task is carried out in the next section.

3. The main theorems

Throughout this section we assume

$$\frac{1}{\lambda_n \pi_n} \sim C n^{\beta-1}, \qquad \pi_n \sim D n^{\gamma-1}, \tag{63}$$

where C and D are fixed positive constants. For ease of exposition the preparations leading to the main result are presented in the form of a series of lemmas, some of which may have independent interest. In lemmas 1, 2 we assume $\gamma > 0$, $\beta + \gamma > 0$, but for lemmas 3, 4, 5, 6 we assume $\beta > 0$, $\gamma > 0$.

LEMMA 1. *For any fixed $r \geqq 1$, as $n \to \infty$*

$$\frac{|Q_n^{(r)}(0)|}{r!} \sim \frac{(CD)^r n^{r(\beta+\gamma)}}{\left[\prod_{k=1}^{r} (k - \gamma + (k-1)\beta)\right] (\beta+\gamma)^r r!} = e_r n^{r(\beta+\gamma)}, \tag{64}$$

where e_r is defined in the obvious manner.

PROOF. The proof is by induction on r based on formula (46) and the trivial asymptotic relations $\sum_1^n k^{a-1} \sim n^a/a$ with $a > 0$. We indicate the proof in the case $r = 1$. For any $\epsilon > 0$ let $M(\epsilon)$ be determined so that for all $n \geqq M(\epsilon)$

$$(1-\epsilon) C n^{\beta-1} \leqq \frac{1}{\lambda_n \pi_n} \leqq (1+\epsilon) C n^{\beta-1} \tag{65}$$

$$\text{and} \quad (1-\epsilon) D n^{\gamma-1} \leqq \pi_n \leqq (1+\epsilon) D n^{\gamma-1}.$$

Now

$$\sum_{k=0}^{n-1} \frac{1}{\lambda_k \pi_k} \sum_{l=0}^{k} \pi_l = \sum_{k=M}^{n-1} \frac{1}{\lambda_k \pi_k} \sum_{l=M}^{k} \pi_l + O\left(\sum_{k=M}^{n-1} \frac{1}{\lambda_k \pi_k}\right). \tag{66}$$

Using the above estimates we obtain

$$\begin{aligned}(1-\epsilon)^2 CD \sum_{k=M}^{n-1} k^{\beta-1} \sum_{l=M}^{k} l^{\gamma-1} &\leqq |Q_n^{(1)}(0)| \\ &\leqq (1+\epsilon)^2 CD \sum_{k=M}^{n-1} k^{\beta-1} \sum_{l=M}^{k} l^{\gamma-1} + O(n^{\beta}),\end{aligned} \tag{67}$$

and from this the assertion for $r = 1$ follows. The cases $r \geqq 2$ are done similarly.

We next obtain a bound on the coefficients of $Q_n(x)$ which is weaker than (64) but valid uniformly in n and r.

LEMMA 2. *For all n and $1 \leqq r \leqq n$*

$$\left|\frac{Q_n^{(r)}(0)}{r!}\right| \leqq \frac{M^{2r}}{\gamma^r \beta^r} \frac{(n+1)^{r(\beta+\gamma)}}{(r!)^2}, \tag{68}$$

where M is an appropriate fixed constant independent of n and r.

PROOF. In view of (63) we may choose M sufficiently large so that

$$\frac{1}{\lambda_n \pi_n} \leqq M(n+1)^{\beta-1} \quad \text{and} \quad \pi_n \leqq M(n+1)^{\gamma-1}, \qquad n \geqq 0. \tag{69}$$

It follows by comparison with an integral of the form $\int_1^n x^a\,dx$ for an appropriate a that

$$\sum_{k=0}^{n-1} \frac{1}{\lambda_k \pi_k} \leqq \frac{M}{\beta}(n+1)^{\beta} \quad \text{and} \quad \sum_{k=0}^{n-1} \pi_k \leqq \frac{M}{\gamma}(n+1)^{\gamma}. \tag{70}$$

Thus

$$|Q_n^{(1)}(0)| = \sum_{k=0}^{n-1} \frac{1}{\lambda_k \pi_k} \sum_{l=0}^{k} \pi_l \leqq \left(\sum_{k=0}^{n-1} \frac{1}{\lambda_k \pi_k}\right)\left(\sum_{l=0}^{n-1} \pi_l\right) \leqq \frac{M^2}{\beta\gamma}(n+1)^{\gamma+\beta}, \tag{71}$$

which is the result for $r = 1$. Using induction on r, we get from (46)

$$\begin{aligned}\left|\frac{Q_n^{(r)}(0)}{r!}\right| &\leqq \sum_{k=0}^{n-1} \frac{1}{\lambda_k \pi_k} \sum_{l=0}^{k} \pi_l \frac{|Q_l^{(r-1)}(0)|}{(r-1)!} \\ &\leqq \frac{M^{2r}}{\gamma^{r-1}\beta^{r-1}[(r-1)!]^2} \sum_{k=0}^{n-1} (k+1)^{r\beta-1} \sum_{l=0}^{n-1} (l+1)^{r\gamma-1} \\ &\leqq \frac{M^{2r}}{\gamma^r \beta^r} \frac{(n+1)^{r(\beta+\gamma)}}{(r!)^2},\end{aligned} \tag{72}$$

which completes the proof.

For each fixed positive integer ν let

$$I_{(\nu)}(s) = \sum_{r=0}^{\nu} e_r s^r, \qquad e_0 = 1,\ e_r = \frac{(CD)^r}{r!(\beta+\gamma)^{2r}\left(\prod_{k=1}^{r}(k-\alpha)\right)}, \tag{73}$$

where

$$\alpha = \frac{\beta}{\beta+\gamma}. \tag{74}$$

LEMMA 3. ($\beta > 0$, $\gamma > 0$). *For each fixed* ν *and constant* $B > 0$

$$\lim_{s\to 0+} s^{\alpha} \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n [I_{(\nu)}(Bn^{\beta+\gamma}s)]^2} = B^{-\alpha} \frac{C}{\beta+\gamma} \int_0^{\infty} \frac{t^{\alpha-1}\, dt}{[I_{(\nu)}(t)]^2}. \tag{75}$$

PROOF. The series in (75) can be expressed in integral form as

$$\int_0^{\infty} \frac{1}{[I_\nu(x^{\beta+\gamma}Bs)]^2}\, d\varphi(x), \tag{76}$$

where

$$\varphi(x) = \sum_{k\leq x} \frac{1}{\lambda_k \pi_k} \sim \frac{C}{\beta} x^{\beta} \qquad \text{as} \quad x \to \infty. \tag{77}$$

Let $y = x^{\beta+\gamma}$ and $\tilde{\varphi}(y) = \varphi(y^{1/(\beta+\gamma)})$ and set $Bs = u$. The integral (76) becomes

$$\int_0^{\infty} \frac{d\tilde{\varphi}(y)}{[I_{(\nu)}(yu)]^2}. \tag{78}$$

The asymptotic growth of $\varphi(x)$ implies $\tilde{\varphi}(y) \sim (C/\beta)y^{\alpha}$ as $y \to \infty$. Integrating by parts for $u > 0$ gives

$$f(u) = \int_0^{\infty} \frac{d\tilde{\varphi}(y)}{[I_{(\nu)}(yu)]^2} = \frac{1}{\lambda_0} + u \int_0^{\infty} \frac{\tilde{\varphi}(y)2[I'_{(\nu)}(yu)]\, dy}{[I_{(\nu)}(yu)]^3}. \tag{79}$$

A direct calculation yields

$$u^{\alpha}f(u) - H(\nu) = \frac{u^{\alpha}}{\lambda_0} + u^{\alpha+1} \int_0^{\infty} \frac{\left[\tilde{\varphi}(y) - \dfrac{C}{\beta} y^{\alpha}\right] 2I'_{(\nu)}(yu)\, dy}{[I_\nu(yu)]^3}, \tag{80}$$

where

$$H(\nu) = \frac{C}{\beta} 2 \int_0^{\infty} \frac{t^{\alpha} I'_{(\nu)}(t)\, dt}{[I_{(\nu)}(t)]^3} = \frac{C\alpha}{\beta} \int_0^{\infty} \frac{t^{\alpha-1}}{[I_{(\nu)}(t)]^2}\, dt = \frac{C}{\beta+\gamma} \int_0^{\infty} \frac{t^{\alpha-1}\, dt}{[I_{(\nu)}(t)]^2}. \tag{81}$$

Splitting the integral on the right in (80) into two parts, one extended over the segment $[0, T]$ and the other over the segment (T, ∞), we easily deduce

$$\overline{\lim_{u\to 0+}} |u^{\alpha}f(u) - H(\nu)| \leq H(\nu) \sup_{t\geq T} \left|\frac{\beta}{C} t^{-\alpha}\tilde{\varphi}(t) - 1\right|. \tag{82}$$

Since T can be made arbitrarily large the lemma follows.

LEMMA 4. ($\beta > 0$, $\gamma > 0$).

$$\overline{\lim}\, s^{\alpha}h(s) \leq \frac{C}{\beta+\gamma} \int_0^{\infty} \frac{t^{\alpha-1}\, dt}{[I_{(\nu)}(t)]^2} = H(\nu), \qquad \alpha = \frac{\beta}{\beta+\gamma}. \tag{83}$$

PROOF. Choose $B < 1$ arbitrary and fixed. By virtue of lemma 1 there exists $N(B)$ so that for all $n \geq N$

$$\min\left(\frac{|Q^{(r)}_{n+1}(0)|}{r!}, \frac{|Q^{(r)}_{n}(0)|}{r!}\right) \geq Be_r n^{r(\gamma+\beta)} \geq B^r e_r n^{r(\gamma+\beta)}, \qquad r = 1, \cdots, \nu. \tag{84}$$

It follows by the definition of $I_{(\nu)}$ that $\min [Q_{n+1}(-s), Q_n(-s)] \geqq I_{(\nu)}(Bn^{\beta+\gamma}s)$ for all $n \geqq N$ and $s > 0$. Consider now

$$(85)\qquad \begin{aligned} s^\alpha h(s) &= s^\alpha \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n Q_n(-s) Q_{n+1}(-s)} \\ &= s^\alpha \sum_{n=0}^{N} \frac{1}{\lambda_n \pi_n Q_n(-s) Q_{n+1}(-s)} + s^\alpha \sum_{n=N+1}^{\infty} \frac{1}{\lambda_n \pi_n Q_n(-s) Q_{n+1}(-s)} \\ &\leqq s^\alpha \sum_{n=0}^{N} \frac{1}{\lambda_n \pi_n Q_n(-s) Q_{n+1}(-s)} + s^\alpha \sum_{n=N+1}^{\infty} \frac{1}{\lambda_n \pi_n [I_{(\nu)}(Bn^{\beta+\gamma}s)]^2} \\ &= s^\alpha h_N(s) + s^\alpha \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n [I_{(\nu)}(Bn^{\beta+\gamma}s)]^2}, \end{aligned}$$

where

$$(86)\qquad h_N(s) = \sum_{n=0}^{N} \frac{1}{\lambda_n \pi_n Q_n(-s) Q_{n+1}(-s)} - \sum_{n=0}^{N} \frac{1}{\lambda_n \pi_n [I_{(\nu)}(Bn^{\beta+\gamma}s)]^2}.$$

Invoking lemma 3, we obtain

$$(87)\qquad \overline{\lim_{s \to 0+}}\, s^\alpha h(s) \leqq B^{-\alpha} \frac{C}{\beta+\gamma} \int_0^\infty \frac{t^{\alpha-1}\, dt}{[I_{(\nu)}(t)]^2}.$$

Since B is arbitrarily less than 1 we now let B approach 1 and (83) is established.

The sequence of constants $H(\nu)$ is decreasing and in the limit as $\nu \to \infty$ we have

$$(88)\qquad \overline{\lim_{s \to 0+}}\, s^\alpha h(s) \leqq \frac{C}{\beta+\gamma} \int_0^\infty \frac{t^{\alpha-1}\, dt}{[I(t)]^2} = H(\infty),$$

where

$$(89)\qquad I(t) = \sum_{r=0}^{\infty} e_r t^r, \qquad c_0 = 1 \quad \text{and} \quad e_r = \frac{(CD)^r}{r!(\beta+\gamma)^{2r}} \frac{1}{\prod_{k=1}^{r} (k-\alpha)}.$$

In order to verify the reverse inequality to that of (88) we introduce the functions

$$(90)\qquad J_{(\nu)}(t) = \sum_{r=0}^{\nu} e_r t^r + \sum_{r=\nu+1}^{\infty} f_r t^r,$$

where e_r is the same as before, $f_r = (M^{2r}/\gamma^r \beta^r)[1/(r!)^2]$ and M is defined according to (68).

A trivial inequality on e_r gives

$$(91)\qquad e_r \leqq \left[\frac{CD}{\gamma(\beta+\gamma)}\right]^r \frac{1}{(r!)^2}.$$

We could have determined the constant M appearing in the upper bound arbitrarily so long as it is sufficiently big and (68) persists. Suppose hereafter that M is chosen satisfying

$$\frac{CD\beta}{\beta+\gamma} \leqq M^2. \tag{92}$$

Then it follows on comparison with (89) that

$$e_r \leqq f_r. \tag{93}$$

LEMMA 5. ($\beta > 0, \gamma > 0$). *For each fixed* ν *and* $B > 0$

$$\lim_{s\to 0+} s^\alpha \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n \{J_{(\nu)}[(n+2)^{\beta+\gamma}Bs]\}^2} = \frac{B^{-\alpha}C}{\beta+\gamma}\int_0^\infty \frac{t^{\alpha-1}\,dt}{[J_{(\nu)}(t)]^2}. \tag{94}$$

The proof is similar to that of lemma 3 and will be omitted.

LEMMA 6. ($\beta > 0, \gamma > 0$).

$$\varliminf_{s\to 0+} s^\alpha h(s) \geqq \frac{C}{\beta+\gamma}\int_0^\infty \frac{t^{\alpha-1}\,dt}{[J_{(\nu)}(t)]^2}. \tag{95}$$

PROOF. Choose $B > 1$ arbitrary and fixed and let ν be a fixed integer. By lemma 2 there exists $N = N(B)$ so that for all $n \geqq N$

$$\max\left(\frac{|Q_{n+1}^{(r)}(0)|}{r!}, \frac{|Q_n^{(r)}(0)|}{r!}\right) \leqq Be_r(n+2)^{r(\beta+\gamma)} \leqq B^r e_r(n+2)^{r(\beta+\gamma)}, \qquad r = 1, 2, \cdots, \nu. \tag{96}$$

On account of lemma 2

$$\frac{|Q_n^{(r)}(0)|}{r!} \leqq f_r(n+1)^{r(\beta+\gamma)} \leqq f_r(n+2)^{r(\beta+\gamma)} \tag{97}$$

for all $0 \leqq r \leqq n$. It follows from (96) and (97) that for $n \geqq N$,

$$\begin{aligned}\max[Q_{n+1}(-s), Q_n(-s)] &\leqq \sum_{r=0}^{\nu} B^r e_r (n+2)^{r(\beta+\gamma)} s^r + \sum_{r=\nu+1}^{\infty} B^r f_r (n+2)^{r(\beta+\gamma)} s^r \\ &= J_{(\nu)}(B(n+2)^{\beta+\gamma}s).\end{aligned} \tag{98}$$

The remainder of the proof is an adaptation of the reasoning in lemma 4 and we shall be brief. We have

$$\begin{aligned} s^\alpha h(s) &= s^\alpha \sum_{n=0}^{\infty} \frac{1}{\lambda_n\pi_n Q_n(-s)Q_{n+1}(-s)} \\ &\geqq s^\alpha \sum_{n=0}^{N} \frac{1}{\lambda_n\pi_n Q_n(-s)Q_{n+1}(-s)} + s^\alpha \sum_{n=N+1}^{\infty} \frac{1}{\lambda_n\pi_n [J_{(\nu)}(B(n+2)^{\beta+\gamma}s)]^2} \\ &= s^\alpha \sum_{n=0}^{\infty} \frac{1}{\lambda_n\pi_n [J_{(\nu)}(B(n+2)^{(\beta+\gamma)}s)]^2} + s^\alpha L_N(s).\end{aligned} \tag{99}$$

It follows by lemma 5 that

$$\varliminf_{s\to 0+} s^\alpha h(s) \geqq \frac{B^{-\alpha}C}{\beta+\gamma}\int_0^\infty \frac{t^{\alpha-1}\,dt}{[J_{(\nu)}(t)]^2}. \tag{100}$$

Since $B > 1$ is arbitrary we may let $B \to 1$ and the desired result follows.

As $\nu \to \infty$, $J_\nu(t) \geqq J_{\nu+1}(t)$ which decreases monotonically pointwise to the function $I(t)$. Allowing $\nu \to \infty$ in (100) shows that

$$\underline{\lim}_{s \to 0+} s^\alpha h(s) \geqq \frac{C}{\beta + \gamma} \int_0^\infty \frac{t^{\alpha-1}\, dt}{[I(t)]^2}. \tag{101}$$

The function $I(t)$ in (89) can be expressed in terms of the modified Bessel function $I_{-\alpha}(x)$ where $\alpha = \beta/(\beta + \gamma)$; in fact

$$I(t) = \Gamma(1 - \alpha) \left[\frac{CDt}{(\beta + \gamma)^2}\right]^{\alpha/2} I_{-\alpha}\left(\frac{2\sqrt{CDt}}{\beta + \gamma}\right). \tag{102}$$

The constant $H(\infty)$ in (88) can be expressed in terms of the integral

$$\int_0^\infty \frac{1}{I^2_{-\alpha}(s)} \frac{ds}{s}, \tag{103}$$

which can be evaluated if we observe that $I_{-\alpha}(x) \int_0^x [1/I_{-\alpha}(s)^2 s]\, ds$ is a second solution of the modified Bessel equation $(0 < \alpha < 1)$. The final result is

$$H(\infty) = \frac{(\beta + \gamma)^{2\alpha-1}}{C^{\alpha-1} D^\alpha} \frac{\Gamma(\alpha)}{\Gamma(1 - \alpha)}. \tag{104}$$

Combining (104), (101), and (88) gives theorem 1.

We are also within easy reach of theorem 2. If $s \geqq 0$, $B > 1$, then for every integer ν inequality (98) gives

$$\limsup_{n \to \infty} Q_n \left(\frac{-s}{(n + 2)^{\beta+\gamma}}\right) \leqq J_{(\nu)}(Bs). \tag{105}$$

Letting $B \to 1$ and afterwards $\nu \to \infty$, we have

$$\limsup_{n \to \infty} Q_n \left[\frac{-s}{(n + 2)^{\beta+\gamma}}\right] \leqq I(s) \tag{106}$$

for $s \geqq 0$. On the other hand if $s \geqq 0$, $B < 1$, then for every integer ν inequality (84) gives

$$\liminf_{n \to \infty} Q_n \left[\frac{-s}{(n + 2)^{\beta+\gamma}}\right] \geqq I_{(\nu)}(Bs). \tag{107}$$

Letting $B \to 1$ and afterwards $\nu \to \infty$ we have

$$\liminf_{n \to \infty} Q_n \left[\frac{-s}{(n + 2)^{\beta+\gamma}}\right] \geqq I(s). \tag{108}$$

Hence $Q_n[-s/(n + 2)^{\beta+\gamma}] \to I(s)$ for $s \geqq 0$. Now if $s \geqq 0$ and $\epsilon > 0$ we have, if n is large enough,

$$Q_n \left[\frac{-s}{(n + 2)^{\beta+\gamma}}\right] \leqq Q_n \left[\frac{-s}{n^{\beta+\gamma}}\right] \leqq Q_n \left[\frac{-(1 + \epsilon)s}{(n + 2)^{\beta+\gamma}}\right] \tag{109}$$

and hence $Q_n(-s/n^{\beta+\gamma}) \to I(s)$ for $s \geqq 0$. In any circle $|s| \leqq R$ of the complex plane the polynomial $Q_n(-s/n^{\beta+\gamma})$ has its maximum modulus at $s = R$ and so by the Vitali theorem the convergence is uniform as asserted in theorem 2.

We close this section by describing two examples where theorems 1 and 2 are applicable.

EXAMPLE 1. *Linear growth.* Let

$$\begin{aligned} \lambda_n &= n + a, & n &\geqq 0, \\ \mu_n &= n + b, & n &\geqq 1,\ \mu_0 = 0. \end{aligned} \tag{110}$$

This describes a model of biological growth where the birth and death rates are proportional to population size and immigration and emigration may occur.

A direct calculation yields

$$\pi_n = \frac{\Gamma(n + a)}{\Gamma(n + 1 + b)} \frac{\Gamma(b + 1)}{\Gamma(a)}. \tag{111}$$

We restrict ourselves first to the case where $a > b$ which ensures that the process is either null recurrent or transient. It follows that

$$\begin{aligned} \pi_n &\sim n^{a-1-b} \frac{\Gamma(b + 1)}{\Gamma(a)}, \\ \frac{1}{\lambda_n \pi_n} &\sim n^{b-a} \frac{\Gamma(a)}{\Gamma(b + 1)}. \end{aligned} \tag{112}$$

In the case $b + 1 > a > b$, the process is null recurrent and theorem 1 is in force. Comparing (112) with (63) leads to the identifications $\beta = b + 1 - a$, $\gamma = a - b$ and therefore $\alpha = b + 1 - a$. It follows by theorem 1 and (7) that the fraction of time in $[0, t]$ that the population size is zero is of order t^{b+1-a} as $t \to \infty$. An extreme case of this example $b = 0$, $a = 1$ fits the assumption of (60). In this situation

$$\pi_n \sim D, \qquad \frac{1}{\lambda_n \pi_n} \sim \frac{C}{n}, \tag{113}$$

where C and D are suitable constants. The normalization constant for occupation time random variable is now $C \log t$ rather than a power of t.

EXAMPLE 2. *Queueing.* Let

$$\lambda_n = \lambda,\ n \geqq 0, \qquad \mu_n = \lambda,\ n \geqq 1, \qquad \mu_0 = 0. \tag{114}$$

This corresponds to a simple queueing phenomenon involving exponential interarrival and exponential service times. The case of main interest in this paper is concerned with the critical case where the rate of arrival equals the rate of service. Trivially, we have

$$\pi_n \sim 1, \qquad \frac{1}{\lambda_n \pi_n} \sim \frac{1}{\lambda}. \tag{115}$$

Comparison with (63) shows that $\beta = \gamma = 1$ and consequently $\alpha = 1/2$. On the basis of theorem 1 and (7) we conclude that the fraction of time the server is free (the process is in state 0) during the time interval $[0, t]$ is of order $\sqrt{t}$ as $t \to \infty$.

4. Extensions

A function $L(t)$ with $0 \leqq t < \infty$, is called slowly varying as $t \to \infty$ if it is continuous and

$$\lim_{t \to \infty} \frac{L(\lambda t)}{L(t)} = 1 \qquad \text{for} \quad \lambda > 0. \tag{116}$$

It has been shown by Karamata [6] that L is slowly varying if and only if it has a representation

$$L(t) = \varphi(t) \exp\left\{\int_1^t \frac{\epsilon(t)}{t}\, dt\right\}, \tag{117}$$

where φ and ϵ are continuous functions such that $\varphi(t) \to c > 0$, $\epsilon(t) \to 0$ as $t \to \infty$. From this it may be deduced that the limit in (116) is uniform in any interval $0 < \lambda_0 \leqq \lambda \leqq \lambda_1 < \infty$. This fact is helpful in deducing the asymptotic relation

$$\sum_{k=1}^{n} k^{a-1} L(k) \sim \frac{n^a L(n)}{a}, \qquad a > 0. \tag{118}$$

The proof of (118) proceeds as follows. We decompose the sum into two segments $[1, [\epsilon n] - 1)$, $([\epsilon n], n)$ where $\epsilon > 0$ is chosen fixed but may be arbitrarily small ($[\epsilon n]$ designates the largest integer preceding ϵn). Clearly

$$\sum_{1}^{[\epsilon n]} k^{\alpha-1} L(k) \leqq \tilde{C} \frac{\epsilon^\alpha n^\alpha}{\alpha} L(n), \tag{119}$$

where $\tilde{C}$ is an appropriate constant. For sufficiently large n, $1 - \delta \leqq L(k)/L(n) \leqq 1 + \delta$ if $[\epsilon n] \leqq k \leqq n$, where δ is a small positive number. It follows that

$$(1 - \delta - \epsilon^\alpha) L(n) \frac{n^\alpha}{\alpha} \leqq \sum_{[\epsilon n]}^{n} k^{\alpha-1} L(k) \leqq (1 + \delta) \frac{L(n)(n+1)^\alpha}{\alpha}. \tag{120}$$

Dividing by $L(n) n^\alpha / \alpha$ we obtain

$$1 - \delta - 2\epsilon^\alpha \leqq \varliminf \alpha \frac{\sum_1^n k^{\alpha-1} L(k)}{n^\alpha L(n)} \leqq \varlimsup \alpha \frac{\sum_1^n k^{\alpha-1} L(k)}{n^\alpha L(n)} \leqq \epsilon^\alpha + 1 + \delta. \tag{121}$$

But $\epsilon > 0$ and $\delta > 0$ are arbitrary and therefore (118) obtains.

From (118) we have, since $a > 0$,

$$\sum_{k=0}^{n} k^{a-1} L(k) \leqq M \frac{(n+1)^a L(n)}{a}, \tag{122}$$

where M is a constant not depending on n.

The results of the previous section may be extended to deal with the more general asymptotic laws

$$\frac{1}{\lambda_n \pi_n} \sim C n^{\beta-1} L(n), \qquad \pi_n \sim D n^{\gamma-1} K(n), \tag{123}$$

where C, D, γ, $\beta + \gamma$ are positive constants and $L(t)$, $K(t)$ are slowly varying as $t \to \infty$. By imitating the analysis of lemmas 1 and 2 we obtain

$$\frac{|Q_n^{(r)}(0)|}{r!} \sim e_r L(n) K(n) n^{r(\beta+\gamma)}, \tag{124}$$

r fixed, with e_r as in (89) and

$$\frac{|Q_n^{(r)}(0)|}{r!} \leqq f_r L(n) K(n) n^{r(\beta+\gamma)}, \tag{125}$$

all n, $1 \leqq r \leqq n$, with f_r as in (90). Continuing to imitate the previous analysis we arrive at the following result.

THEOREM 3. *If* (123) *holds with* $K(n) = 1/L(n)$, $\beta > 0$, $\gamma > 0$, *then as* $s \to 0+$

$$h(s) \sim H(\infty) s^{-\alpha} L(s^{-1/\beta+\gamma}), \tag{126}$$

where $\alpha = \beta/(\beta + \gamma)$ *and* $H(\infty)$ *is given by* (104).

In the proof one encounters the integral

$$\int_0^\infty \frac{uI'(uy)}{I^3(uy)} \frac{y^\alpha}{\beta} L(y^{1/\beta+\gamma})\, dy \tag{127}$$
$$= \frac{u^{-\alpha}}{\beta} \int_0^\infty \frac{I'(t)}{I^3(t)} t^\alpha \frac{L\left[\left(\frac{t}{u}\right)^{1/\beta+\gamma}\right]}{L\left[\left(\frac{1}{u}\right)^{1/\beta+\gamma}\right]} L\left[\left(\frac{1}{u}\right)^{1/\beta+\gamma}\right] dt.$$

Using the properties of slowly varying functions it follows by dominated convergence that as $u \to 0+$ this is asymptotic to

$$\frac{u^{-\alpha}}{\beta} L(u^{-1/\beta+\gamma}) \int_0^\infty \frac{I'(t)}{I^3(t)} t^\alpha\, dt. \tag{128}$$

In a simpler fashion we also obtain

THEOREM 4. *If* (123) *holds with* $\gamma > 0$, $\beta + \gamma > 0$, *then*

$$\lim_{n\to\infty} Q_n\left[\frac{-s}{n^{\beta+\gamma}L(n)K(n)}\right] = I(s), \tag{129}$$

where $I(s)$ *is given by* (102) *and the convergence is uniform in every bounded region of the complex variable* s.

The detailed proofs of theorems 3 and 4 are omitted.

EXAMPLES. Consider the generalized linear growth model with

$$\begin{aligned} \lambda_n &= n + a + o(1), & n &\geqq 0, \\ \mu_n &= n + b + o(1), & n &\geqq 1,\ \mu_0 = 0, \end{aligned} \tag{130}$$

where $b + 1 > a > b$, $a > 0$. We then have

(131)
$$\begin{aligned} \log \pi_n &= -\log n + \log \lambda_0 + \sum_{k=1}^{n-1} \log\left[1 + \frac{a}{k} + \frac{o(1)}{k}\right] - \sum_{k=1}^{n} \log\left[1 + \frac{b}{k} + \frac{o(1)}{k}\right] \\ &= (a - b - 1)\log n + \gamma_n + \sum_{k=1}^{n} \frac{\epsilon_k}{k}, \end{aligned}$$

where γ_n has a finite limit as $n \to \infty$ and $\epsilon_k = (\lambda_k - a) - (\mu_k - b)$. Hence, taking exponentials, $\pi_n = n^{a-b-1}L(n)$, where

(132) $$L(n) = e^{\gamma_n} \exp \sum_{k=1}^{n} \frac{\epsilon_k}{k}.$$

If $\sum_1^\infty \epsilon_k/k$ converges we have the case of the previous section but if this series diverges then $L(n)$ can be extended to a slowly varying function $L(t)$ which may $\to 0$ or $+\infty$ as $n \to \infty$. Thus

(133) $$\pi_n \sim n^{a-b-1}L(n), \qquad \frac{1}{\lambda_n \pi_n} \sim n^{b-a}K(n),$$

where $L(t)$ and $K(t) = 1/L(t)$ are slowly varying.

Finally we discuss a set of asymptotic growth laws of exponential character for which an analogue of theorem 4 holds. This result is typical of a large class of limit theorems and we do not attempt to exhaust all the possibilities. The method is that of theorem 4 and is quite general. It is emphasized that theorem 3 is not correct under the conditions of theorem 5.

THEOREM 5.

(i) *Let*

(134) $$\pi_n = C\rho^n n^\beta L(n), \qquad \frac{1}{\lambda_n \pi_n} = D \frac{1}{\rho^n} n^{-\beta-1} \frac{1}{L(n)}, \qquad n \geqq 1,$$

where $\rho > 1$, $L(n)$ *is slowly varying as* $n \to \infty$ *and* β *is real. Then*

(135) $$Q_n\left(\frac{-s}{CD \log n}\right) \to \exp\left[\rho s/(\rho - 1)\right],$$

and the convergence is uniform in every bounded region of the complex plane.

(ii) *Let*

(136) $$\pi_n \sim C\rho^n n^\beta L(n), \qquad \frac{1}{\lambda_n \pi_n} \sim D \frac{n^{-\beta-\delta}}{L(n)\rho^n},$$

where $\rho > 1$, $L(n)$ *is slowly varying* $0 < \delta < 1$ *and* β *is real. Then*

(137) $$Q_n\left(\frac{-s}{CDn^{1-\delta}}\right) \to \exp\left[\frac{\rho s}{(\rho - 1)(1 - \delta)}\right]$$

uniformly on every bounded region of the complex plane.

To illustrate this theorem consider a linear growth process with

(138) $$\lambda_n = \lambda n + a, \qquad \mu_n = \mu n + b,$$

where $\lambda > \mu$, a and b are real. A simple calculation gives

(139) $$\pi_n \sim C\rho^n n^{\gamma-\delta-1}, \qquad \frac{1}{\lambda_n \pi_n} \sim \frac{D}{\rho^n} n^{\delta-\gamma},$$

where

(140) $$\rho = \frac{\lambda}{\mu} > 1, \qquad \gamma = \frac{a}{\lambda}, \qquad \delta = \frac{b}{\mu}, \qquad C = \frac{\Gamma\left(\frac{b}{\mu} + 1\right)}{\Gamma\left(\frac{a}{\lambda}\right)}, \qquad D = \frac{\Gamma\left(\frac{a}{\lambda}\right)}{\lambda\Gamma\left(\frac{b}{\mu} + 1\right)}.$$

5. The maximum random variables

We shall develop a limit law for the random variables

$$M(t) = \max_{0 \leqq u \leqq t} X(u), \tag{141}$$

where $X(t)$ is the birth and death process with $X(0) = 0$, whose infinitesimal parameters obey the asymptotic growth law (63) or more generally (123). We will treat only the case (63). In analyzing (141) we use a familiar duality principle. Let the random variable $T_{i,j}$ be the first passage time for a particle starting in state i to reach state j. Since the path functions are continuous [10] (transitions occur only to neighboring states) we see that

$$T_{i,j} = T_{i,i+1} + T_{i+1,i+2} + \cdots + T_{j-1,j}, \qquad j > i. \tag{142}$$

The Laplace transform of the distribution of $T_{i,j}$ is [10]

$$\mathcal{L}(T_{i,j}) = \frac{Q_i(-s)}{Q_j(-s)}, \qquad s > 0. \tag{143}$$

The Laplace transform of the distribution of $T_{i,n}/n^{\beta+\gamma}$ is then

$$\frac{Q_i\left(\frac{-s}{n^{\beta+\gamma}}\right)}{Q_n\left(\frac{-s}{n^{\beta+\gamma}}\right)}. \tag{144}$$

Appealing to theorem 2, we deduce that the distribution function $F_n(s)$ of the random variables $T_{i,n}/n^{\beta+\gamma}$ converges to a distribution function $F(x)$ of a random variable whose Laplace transform is $\varphi(s) = 1/I(s)$, where $I(s)$ is given in (89). We also note that $F(x)$ is independent of the initial state i. From the explicit expression for $\mathcal{L}(F)$ it follows easily that $F(x)$ is absolutely continuous, and hence $F_n(x) \rightarrow F(x)$ uniformly on $0 \leqq x < \infty$. We convert the limit statement regarding the variables $T_{0,n}$ into one involving $M(t)$ with the aid of the relation

$$P\{T_{0,n} \leqq u\} = P\{M(u) \geqq n\}. \tag{145}$$

From (145) follows $F_n(u) = P\{M(n^{\beta+\gamma}u) \geqq n\}$. Let $t = n^{\beta+\gamma}u$, $n = (t/u)^{1/(\beta+\gamma)}$. Then we have

$$F(u) = \lim_{t \to \infty} P\left\{\frac{M(t)}{t^{1/\beta+\gamma}} \geqq \frac{1}{u^{1/(\beta+\gamma)}}\right\} \tag{146}$$

or equivalently

$$\lim_{t \to \infty} P\left\{\frac{M(t)}{t^{1/(\beta+\gamma)}} \leqq x\right\} = 1 - F\left(\frac{1}{x^{\beta+\gamma}}\right). \tag{147}$$

Although this limit law is "explicitly known" in the sense that $\mathcal{L}(F) = 1/I(s)$, it is complex and unwieldy. We next show how to compute the asymptotic moments of $M(t)$, illustrating the method for the first moment.

Let

$$U(t) = \sum_{n=1}^{\infty} P\{T_{0,n} \leqq t\}. \tag{148}$$

Since the right member of

$$\sum_{n=1}^{\infty} \int_0^{\infty} e^{-st} P\{T_{0,n} \leqq t\}\, dt = \frac{1}{s} \sum_{n=1}^{\infty} \frac{1}{Q_n(-s)} \tag{149}$$

converges for every $s > 0$, we see that $U(t) < \infty$ for almost all t. But $U(t)$ is, like each term in (148), a nondecreasing function of t. Hence (148) converges for all $t \geqq 0$. Now for fixed t each term of (148) is a nonnegative decreasing function of n, so by partial summation

$$\begin{aligned} U(t) &= \sum_{n=1}^{\infty} n[P\{T_{0,n} \leqq t\} - P\{T_{0,n+1} \leqq t\}] \\ &= \sum_{n=0}^{\infty} nP\{M(t) = n\} = E[M(t)]. \end{aligned} \tag{150}$$

Now following the method of section 2 we get

$$\sum_{n=1}^{\infty} \frac{1}{Q_n(-s)} \sim \frac{1}{(\beta+\gamma)s^{1/(\beta+\gamma)}} \int_0^{\infty} \frac{t^{1/(\beta+\gamma)}}{I(t)} \frac{dt}{t} \tag{151}$$

and hence

$$\int_0^{\infty} e^{-st} E(M(t))\, dt \sim \frac{K_0}{\beta+\gamma} s^{-1-1/(\beta+\gamma)}, \qquad s \to 0, \tag{152}$$

where K_0 is the integral in (151). Since $E[M(t)]$ is positive and increasing we may use Karamata's theorem to get from (152)

$$E[M(t)] \sim \frac{K_0}{(\beta+\gamma)\Gamma\left(1 + \dfrac{1}{\beta+\gamma}\right)} t^{1/(\beta+\gamma)}, \qquad t \to \infty. \tag{153}$$

Calculation of higher asymptotic moments leads to similar but more tedious calculations.

We close this section with

EXAMPLE 1. Let $\lambda_n = a$ with $n \geqq 0$, and let $\mu_n = a$, with $n \geqq 1$, and $\mu_0 = 0$. We found earlier (the end of section 3) that in this case $\beta = \gamma = 1$. Then

$$\lim_{n\to\infty} \mathcal{L}\left(\frac{T_{0n}}{n^2}\right) = \frac{1}{\cosh \sqrt{t/a}}. \tag{154}$$

The inversion of $(\cosh \sqrt{t/a})^{-1}$ gives a theta function distribution $F(x)$ whose density is

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{a\pi}\, x^{3/2}} \displaystyle\sum_{n=0}^{\infty} (-1)^n (2n+1) \exp[-(2n+1)^2/4ax] & \text{for } x \geqq 0 \\ 0 & \text{for } x < 0. \end{cases} \tag{155}$$

The density function $g(x)$ of the limit distribution of $M(t)/\sqrt{t}$ is

$$g(x) = \frac{2}{x^3} f\left(\frac{1}{x^2}\right). \tag{156}$$

6. Other applications: sojourn time in a half interval

We pointed out in the introduction that knowing that $h(s)$ possesses the form (6) leads to numerous consequences and insights pertaining to the nature of various distributions of functionals on the process. In particular, under the conditions (63) of section 3 we may apply the Dynkin-Lamperti theorem referred to in (31). For example, if we consider the queueing model of section 3 we conclude that the time since the server was last free grows like t and the finer knowledge of this distribution is expressed in (31).

Another direction of applications of (6) has to do with the occupation time of an infinite set of states. Consider a birth and death process whose state space is now the full set of integers, $-\infty < n < \infty$. Let λ_n, μ_n and λ'_n, μ'_n denote the birth and death rates for $n \geqq 0$ and $n < 0$ respectively. We assume these are all positive.

Let the random variable $\xi(t)$ be the time during the interval $(0, t)$ for which $X(t) \geqq 0$. In other words, $\xi(t)$ is the occupation time for the set of nonnegative states during $(0, t)$. Similarly, let $\eta(t)$ denote the occupation time during the interval $[0, t)$ for which $X(t) < 0$. Obviously $\xi(t) + \eta(t) = t$. We assume that

$$\begin{aligned} &\frac{1}{\lambda_n \pi_n} \sim C n^{\beta - 1}, \qquad \pi_n \sim D n^{\gamma - 1}, \qquad && n \geqq 0, \\ &\frac{1}{\lambda'_{-n} \pi'_{-n}} \sim C' n^{\beta' - 1}, \qquad \pi'_{-n} \sim D' n^{\gamma' - 1}, \qquad && n > 0, \end{aligned} \tag{157}$$

where $\pi_0 = 1$ and

$$\pi_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}, \qquad \pi_{-n} = \frac{\mu'_1 \mu'_2 \cdots \mu'_n}{\lambda'_2 \lambda'_3 \cdots \lambda'_{n+1}}. \tag{158}$$

We could deal equally well with more general rates of growth of type (123) at the expense of greater technical complication with no essential new ideas.

It will be convenient to consider an alternative formulation. Let $U(t)$ represent a stochastic process with two possible states A and B. For definiteness, let $U(0) = A$.

Denote by $\xi_1, \eta_1, \xi_2, \eta_2, \cdots$ the successive sojourn times that $U(t)$ spends in states A and B respectively. We suppose $\{\xi_i\}$ and $\{\eta_i\}$ are independent identically distributed positive random variables. Now let $\tilde{\xi}(t)$ and $\tilde{\eta}(t)$ denote the total sojourn times spent in A and B respectively during the time interval $(0, t)$. Takács in [14] develops several limit laws for $\xi(t)$ and $\tilde{\eta}(t)$ under the conditions that $\{\xi_i\}$ and $\{\eta_j\}$ belong to a domain of attraction of respective stable laws.

We now show that the random variables $\xi(t)$ and $\eta(t)$ can be identified with a family of random variables $\tilde{\xi}(t)$ and $\tilde{\eta}(t)$ arising in the manner of the pre-

ceding paragraph. In fact, let $\{\xi_i\}$ denote independent observations on the first passage time from state 0 to -1, and similarly let $\{\eta_i\}$ denote independent observations of the first passage time from state -1 to 0. As the process $U(t)$ unfolds, we observe a sojourn time spent on the nonnegative axis of length ξ_1, then a time η_1 spent on the negative axis, etc. We associate with the process, at hand, a new stochastic process $U(t)$ of two states A and B by the definition $U(t) = A$ if and only if $X(t) \geqq 0$, $U(t) = B$ if and only if $X(t) < 0$. It is manifestly evident that $\tilde{\xi}(t) = \xi(t)$ and $\tilde{\eta}(t) = \eta(t)$ as claimed.

Let $F_{0,-1}(x)$ denote the first passage distribution from state 0 to state -1. It follows easily from (18) that

$$s \int_0^\infty e^{-sx}[1 - F_{0,-1}(x)]\,dx = \frac{c_0}{h(s)} + c_1, \tag{159}$$

where $h(s)$ has the properties of (6). This implies as noted in connection with (26) by virtue of the Karamata-Tauberian theorem that ξ_i belongs to the domain of attraction of a stable law of index $\alpha = \beta/(\beta + \gamma)$. Similarly, we deduce that $F_{-1,0}(x)$, the first passage distribution from state -1, to state 0, belongs to the domain of attraction of a stable law of index $\alpha' = \beta'/(\beta' + \gamma')$.

We now have all the ingredients with which to apply the results of Takâcs. These yield various limit laws for $\xi(t)$ and $\eta(t)$. Specifically, under the assumptions (157) and suppose for simplicity $\alpha = \alpha'$, then

$$\lim_{t\to\infty} P\left\{\frac{\xi(t)}{t} \leqq x\right\} = H(x) \tag{160}$$

and $H(x)$ is the distribution function of a random variable $\zeta = B_2\theta/(A_2\mathfrak{X} + B_2\theta)$, where θ and $\mathfrak{X}$ are independent random variables each of whose distribution functions is a stable law of index α, and A_2, B_2 are appropriate positive constants determined from $F_{-1,0}(x)$ and $F_{0,-1}(x)$.

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